INFLUENCE OF A HYDRODYNAMIC FIELD ON LAMINAR FLAME STABILITY

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The hydrodynamic stability of a plane flame front has been studied by L. D. Landau in [1], where it was shown that it is absolutely unstable.

The aim of this note is to clarify the influence of the hydrodynamic field curvature on flame stability. Flame stability is analyzed within the framework of Landau's theory, in which the flame front is represented by a surface on which the velocity, density, and temperature values experience discontinuities. Viscosity, diffusion, and heat conduction are neglected. The front moves at a given constant velocity relative to the gas. The gas is assumed to be incompressible in front of and behind the front.

It is shown that fields exist which will both stabilize and destabilize the gas. A cylindrical flame formed by a concentrated source of given intensity is examined (two-dimensional problem). Flame stability is studied for the case of a perturbed flame front. It is shown that in this case, the hydrodynamic field has the effect of stabilizing the flame. For the first perturbation harmonics, the destabilizing effect of gas expansion appears to be relatively weak compared to the stabilizing effect of the velocity field. The first perturbation harmonics attenuate. The destabilizing effect of the velocity field is demonstrated by an example in which the radial flow of the fresh mixture is applied externally, and there exists a concentrated sink flow for the combustion products.

§1. The hydrodynamic pattern of an unperturbed cylindrical flame, in the polar coordinates r and φ has the form

$$v_{r1}^\circ = \frac{Q}{2\pi\rho_1 r},$$

$$\frac{\partial P_1^\circ}{\partial r} = \frac{Q}{4\pi^2\rho_1 r^3}\left(0 \leqslant r \leqslant \frac{Q}{2\pi\rho_1 S_1} = R\right)$$

$$v_{r2}^\circ = \frac{Q}{2\pi\rho_2 r}, \quad \frac{\partial P_2^\circ}{\partial r} = \frac{Q}{4\pi^2\rho_2 r^3} \quad (R \leqslant r). \tag{1.1}$$

where v_r is the radial velocity component of the gas, S is the flame velocity relative to the gas, ρ is the gas density, P is the pressure in the gas, Q is the intensity of the fresh-mixture source, and R is the radius of the unperturbed flame front (all the fresh-mixture parameters are denoted by subscript 1, while subscript 2 denotes the combustion-product parameters).

The stability analysis will be based on the equation for the stream function Ψ

$$\frac{\partial\Delta\Psi}{\partial t} + \frac{1}{r}\frac{\partial\Psi}{\partial\varphi}\frac{\partial\Delta\Psi}{\partial r} - \frac{1}{r}\frac{\partial\Psi}{\partial r}\frac{\partial\Delta\Psi}{\partial\varphi} = 0$$

$$\left(v_r = \frac{1}{r}\frac{\partial\Psi}{\partial\varphi}, \quad v_\varphi = -\frac{\partial\Psi}{\partial r}\right). \tag{1.2}$$

The Euler equation provides a relation between Ψ and P;

$$\frac{\partial^2\Psi}{\partial t\,\partial r} + \frac{1}{r}\frac{\partial\Psi}{\partial r}\frac{\partial^2\Psi}{\partial r^2} -$$

$$-\frac{1}{r}\frac{\partial\Psi}{\partial r}\frac{\partial^2\Psi}{\partial r\,\partial\varphi} + \frac{1}{r}\frac{\partial\Psi}{\partial r}\frac{\partial\Psi}{\partial\varphi} = \frac{1}{\rho r}\frac{\partial P}{\partial\varphi}. \tag{1.3}$$

We set $\Psi = \Psi^\circ + \psi$, $P = P^\circ + p$ (where Ψ°, P° correspond to the unperturbed flame), and assume that

$$|\psi| \ll |\Psi^\circ|, \qquad |p| \ll |P^\circ|. \tag{1.4}$$

Using (1.1), (1.4), we linearized Eqs. (1.2) and (1.3)

$$\frac{\partial\Delta\psi}{\partial t} + \frac{\varkappa}{r}\frac{\partial\Delta\psi}{\partial r} = 0 \qquad \left(\varkappa = \frac{Q}{2\pi\rho}\right) \tag{1.5}$$

$$\frac{\partial^2\psi}{\partial t\,\partial r} + \frac{\varkappa}{r}\frac{\partial^2\psi}{\partial r^2} + \frac{\varkappa}{r^2}\frac{\partial\psi}{\partial r} = \frac{1}{\rho r}\frac{\partial p}{\partial\varphi}. \tag{1.6}$$

From (1.5), we have

$$\Delta\psi = F(t - r^2/2\varkappa, \varphi), \tag{1.7}$$

which for the initial time is

$$\Delta\psi = F(t - r^2/2\varkappa, \varphi) = 0. \tag{1.8}$$

Assuming that the fresh-mixture source is not a vortex source, and taking (1.8) into account, we find that $\Delta\psi = 0$ for $t \geq 0$, $0 \leq r \leq R$ and $r \geq (2\varkappa t + R^2)^{1/2} = D(t)$. Hence, in linear approximation, the flow will be potential only beyond the ring $R \leq r \leq D(t)$.

Thus, a perturbation leads to the separation of a vortex discontinuity surface from the flame front. Since the velocity of this surface relative to the gas is zero, the tangential velocity component can have a discontinuity $w(\varphi, t)$ at $r = D(t)$. Since at the initial time, only the front is perturbed, we set

$$w(\varphi, 0) = 0. \tag{1.9}$$

§2. We write the linearized conditions for mass-flow and momentum-component conservation at the flame front, together with the conditions for the constancy of the flame front velocity relative to the gas. For r = R, we have:

mass flow continuity

$$S_1\rho_1 = S_2\rho_2; \tag{2.1}$$

normal momentum component continuity

$$p_1 - p_2 = \Lambda R^{-3}\rho_1\varkappa_1(\varkappa_2 - \varkappa_1) \quad (|\Lambda| \ll R); \tag{2.2}$$

tangential momentum component continuity

$$\frac{\partial}{\partial r}(\psi_2 - \psi_1) = \frac{1}{R^3}\frac{\partial\Lambda}{\partial\varphi}(\varkappa_2 - \varkappa_1); \tag{2.3}$$

and constancy of the normal front velocity relative to the gas

$$\frac{\partial\Lambda}{\partial t} + \frac{\varkappa_1}{R^2}\Lambda = \frac{1}{R}\frac{\partial\psi_1}{\partial\varphi},$$

$$\frac{\partial}{\partial\varphi}(\psi_1 - \psi_2) = \frac{\Lambda}{R}(\varkappa_1 - \varkappa_2). \tag{2.4}$$

Here Λ is the perturbation of the front.

The linearized conservation conditions on the vortex discontinuity surface and the impenetrability conditions of the surface for the gas, at r = D(t), are:

continuity of the normal momentum component

$$p_2 - p_3 = 0; \qquad (2.5)$$

impenetrability of the vortex discontinuity surface

$$\frac{\partial}{\partial \varphi}(\psi_2 - \psi_3) = 0 \qquad (2.6)$$

and possibility of a discontinuity of the tangential velocity component

$$\frac{\partial}{\partial r}(\psi_2 - \psi_3) = w(\varphi, t) \qquad (2.7)$$

(the combustion product parameters in the region $r \geq D$ are denoted by the subscript 3.)

§3. Let us show that the radial field of a concentrated source has a stabilizing effect on the flame front. To this end, we assume that thermal expansion is absent, i.e., $\rho_1 = \rho_2 = \rho$; $S_1 = S_2 = S$. Under these conditions, a perturbation of the flame front will not lead to a disturbance of the velocity and the pressure.

From the first equation in (2.4), we have a perturbation equation for the front

$$\frac{\partial \Lambda}{\partial t} + \frac{\varkappa}{R^2}\Lambda = 0. \qquad (3.1)$$

From here, it is seen that the flame front is absolutely stable.

We shall show that the hydrodynamic field of a concentrated sink flow has a destabilizing effect on the front. Indeed, assuming $\rho_1 = \rho_2 = \rho$, $S_1 = S_2 = S$, in the same manner as above, we get

$$\frac{\partial \Lambda}{\partial t} - \frac{\varkappa}{R^2}\Lambda = 0.$$

Thus, in this case, the flame front is absolutely unstable. If, however, $\rho_1 > \rho_2$—where ρ_2 is the combustion-product density—the destabilizing effect of thermal expansion will add up with the destabilizing effect of the radial velocity field of the undisturbed flame, thereby increasing the instability of the front.

§4. We expand the functions ψ, p, Λ, w, and F into a Fourier series in φ. The conditions (2.2)–(2.7) reduce to the conditions for the Fourier components (the subscript k denotes the k-th Fourier component)

$$p_{1k} - p_{2k} = \frac{\Lambda_k}{R^3}\rho_1\varkappa_1(\varkappa_2 - \varkappa_1),$$

$$\frac{\partial}{\partial r}(\psi_{2k} - \psi_{1k}) = \frac{ik}{R^3}(\varkappa_2 - \varkappa_1),$$

$$\frac{\partial \Lambda_k}{\partial t} + \frac{\varkappa_1}{R^2}\Lambda_k = \frac{ik}{R}\psi_{1k},$$

$$ik(\psi_1 - \psi_2) = \frac{\Lambda_k}{R}(\varkappa_1 - \varkappa_2), \quad p_{2k} - p_{3k} = 0,$$

$$\psi_{2k} - \psi_{3k} = 0, \ \frac{\partial}{\partial r}(\psi_{2k} - \psi_{3k}) = w_k(t). \qquad (4.1)$$

Equations (1.6) and (1.7) reduce to the equations for the Fourier components

$$\frac{\partial^2 \psi_k}{\partial t\, \partial r} + \frac{\varkappa}{r}\frac{\partial^2 \psi}{\partial r^2} + \frac{\varkappa}{r^2}\frac{\partial \psi_k}{\partial r} = \frac{ik}{\rho r}p_k,$$

$$\frac{\partial^2 \psi_{1k}}{\partial r^2} + \frac{1}{r}\frac{\partial \psi_{1k}}{\partial r} - \frac{k^2}{r^2}\psi_{1k} = 0,$$

$$\frac{\partial^2 \psi_{2k}}{\partial r^2} + \frac{1}{r}\frac{\partial \psi_{2k}}{\partial r} - \frac{k^2}{r^2}\psi_{2k} = F_k,$$

$$\frac{\partial^2 \psi_{3k}}{\partial r^2} + \frac{1}{r}\frac{\partial \psi_{3k}}{\partial r} - \frac{k^2}{r^2}\psi_{3k} = 0. \qquad (4.2)$$

Making use of (4.1.2), (4.1.4), and (4.2.1)–(4.2.3), we transform condition (4.1.1) to

$$\frac{\partial^2}{\partial t\, \partial r}(\rho_1\psi_{1k} - \rho_2\psi_{2k}) = \frac{\rho_1\varkappa_1}{R}F_k\left(t - \frac{R^2}{2\varkappa^2}\right). \qquad (4.3)$$

In the same manner, condition (4.1.5) converts to

$$\varkappa_1\rho_1 F_k\left(t - \frac{D^2}{2\varkappa_2}\right) = D\rho_2\frac{dw_k}{dt}. \qquad (4.4)$$

Here, however, $F_k(t - D^2/2\varkappa_2) = F_k(-R^2/2\varkappa_2) = 0$ by virtue of (1.8).

Hence, (4.1.5) is equivalent to

$$dw_k \,/\, dt = 0.$$

From here, according to (1.9), we have $w_k \equiv 0$. Thus, the initial condition (1.9) ensures the continuity of the tangential velocity component at the surface r = = D(t). Condition (4.1.7) takes the form

$$\frac{\partial}{\partial r}(\psi_{2k} - \psi_{3k}) = 0. \qquad (4.5)$$

Now (4.1.5) is a corollary of (4.5) and, therefore, for r = D(t), it is sufficient to demand that only (4.1.6) and (4.5) be satisfied.

§5. The case k = 0, which corresponds to one-dimensional perturbations of the front, leads to equation (3.1). This means that for k = 0, the front is stable.

For k = 1, 2, ... it is convenient to pass to dimensionless variables with the aid of the following formulas:

$$r = R + R\xi \,/\, k, \qquad t = R\tau \,/\, S_2 k,$$

$$\rho_2 = \varepsilon\rho_1, \quad i\Lambda_k = R\vartheta(\tau)/k,$$

$$\psi_k = RS_2\omega(\tau, \xi) \,/\, k,$$

$$RF_k = S_2 k\,\theta(\tau - \xi - \xi^2 / 2k)$$

Conditions (4.1.1)–(4.1.4), (4.1.6), and (4.5) take the form

$$\frac{\partial^2}{\partial \xi\, \partial \tau}(\omega_1 - \varepsilon\omega_2) = \varepsilon\theta, \quad \frac{\partial}{\partial \xi}(\omega_2 - \omega_1) = \vartheta(1 - \varepsilon),$$

$$\frac{\partial \vartheta}{\partial \tau} + \frac{\varepsilon}{k}\vartheta + \omega_1 = 0,$$

$$\omega_1 - \omega_3 + \frac{1+\varepsilon}{k}\vartheta = 0 \quad (\xi = 0),$$

$$\omega_2 - \omega_3 = 0, \quad \frac{d}{d\xi}(\omega_2 - \omega_3) = 0,$$

$$(\xi = k(\sqrt{1 + 2\tau/k} - 1) = d). \qquad (5.1)$$

Using dimensionless variables, from (4.2.2)–(4.2.4) we get

$$\omega_1 = C_1(\tau)(1 + \xi / k)^k, \qquad (5.2)$$

$$\omega_2 = C_2(\tau)(1+\xi/k)^{-k} + C_3(\tau)(1+\xi/k)^{k}) +$$

$$+\frac{1}{2}\int_0^{\xi} \theta(\eta,\tau)\left(1+\frac{\eta}{\tau}\right)\left[\left(\frac{\xi+k}{\eta+k}\right)^k - \left(\frac{\xi+k}{\eta+k}\right)^{-k}\right]d\eta,$$

$$\omega_3 = C_4(\tau)(1+\xi/k)^{-k}. \qquad (5.2)$$
(cont'd)

Here, the condition $|\psi_1| < \infty$ for $r = 0$, $\psi_3 \to 0$ for $r \to \infty$ is taken into account.

From equations (5.1) and (5.2), one obtains a system of equations for ϑ and θ

$$a\vartheta'' + b\vartheta' + \theta = 0, \quad 2\vartheta' + c\vartheta =$$

$$= \int_0^{\tau} \theta(z)\left(1+\frac{2}{k}(\tau - z)\right)^{-1/2 k} dz,$$

$$a = \frac{1-\varepsilon}{\varepsilon}, \quad b = \frac{(1-\varepsilon)(1+k)}{k},$$

$$k_0 = \frac{1+\varepsilon}{1-\varepsilon}, \quad c = \frac{(1-\varepsilon)(k_0-k)}{k}. \qquad (5.3)$$

Let us examine the case $k \to \infty$, where $(1 + 2(\tau - z)/k)^{-(1/2)k} \to e^{z-\tau}$

If here $Q \to \infty$, so that $Q/k < \infty$, we arrive at the stability problem for a plane flame front, investigated by L. D. Landau [1] by the method of wave equations.

For $k \to \infty$, system (5.3) can be reduced to a single equation for ϑ

$$(1+\varepsilon)\vartheta'' + 2\varepsilon\vartheta' + \varepsilon(\varepsilon - 1)\vartheta = 0.$$

We set $\vartheta(0) = \vartheta_0$, then from (5.3.2) we have $\vartheta'(0) = \vartheta_0(1-\varepsilon)/2$. It follows that the plane flame front is absolutely unstable.

The solution of the equation obtained coincides asymptotically with Landau's solution [1].

§6. For further analysis, it is convenient to transform system (5.3) to

$$\vartheta' = -\frac{\vartheta_0 c}{2}\exp\frac{-b\tau}{a} - \frac{1}{a}\int_0^{\tau}\theta(z)\exp\frac{-b(\tau-z)}{a}dz \qquad (6.1)$$

$$\theta = A\theta \equiv \int_0^{\tau}\theta(z)G(\tau - z)dz +$$

$$+\frac{\vartheta_0 c(2b-ac)}{2(2+a)}\exp\frac{-b\tau}{a}$$

$$\vartheta_0 = \vartheta(0), \qquad G(z) = \frac{a}{a+2}\left(1+\frac{2}{k}z\right)^{-1/2 k - 1} +$$

$$+\frac{2b-ac}{a(2+a)}\exp\frac{-bz}{a}. \qquad (6.2)$$

We note that

$$2b - ac = (kk_0 - 1)/\varepsilon kk_0 \geqslant 0 \quad \text{for} \quad k \geqslant 1.$$

In the following we consider $\vartheta_0 \geq 0$, without loss of generality.

We prove that $\vartheta' \geq 0$ for $k > k_0$ $(c < 0)$. To this end, as can be seen from (6.1), it is sufficient to demonstrate that $\theta \leq 0$. Let us examine a set of functions $\{u(\tau)\}$, such that $-\Omega e^{\gamma\tau} \leq u(\tau) \leq 0$ (Ω, γ are positive constants). We show that for sufficiently large Ω and γ, the operator A maps this set of functions into itself. To this end it is sufficient to show that

$$\Omega e^{\gamma\tau}\left(\int_0^{\tau} G(z)e^{-\gamma z}dz - 1\right) - \frac{\vartheta_0 c(2b-ac)}{2(2+a)} \leqslant 0.$$

This inequality holds a fortiori for sufficiently large Ω and γ. Thus, $-\Omega e^{\gamma\tau} \leq Au \leq 0$ when $-\Omega e^{\gamma\tau} \leq u \leq 0$. From here, according to the Leray-Schauder theorem [2], there exists a $\theta(\tau) \leq 0$ such that $A\theta = \theta$.

In the same manner, one can prove that $\vartheta' \leq 0$ for $k < k_0$ $(c > 0)$. In this case, θ must be sought among a set of functions $\{u(\tau)\}$, such that

$$0 \leqslant u(\tau) \leqslant \Omega \qquad (\Omega = \text{const}).$$

In order that the operator A maps this set into itself, it is sufficient to satisfy the inequality

$$\Omega\left(\int_0^{\tau} G(z)dz - 1\right) + \frac{\vartheta_0 c(2b-ac)}{2(2+a)} \leqslant 0,$$

which is the case at sufficiently large Ω, since

$$\int_0^{\tau} G(z)dz \leqslant 1 - \frac{ac}{b(2+a)} \leqslant 1.$$

On the basis of the aforesaid and (5.3.2), it follows directly that $\vartheta \geq 0$ for $k < k_0$.

§7. We show that $\vartheta \to \infty$ for $\tau \to \infty$, $k > k_0$ and that $\vartheta \to 0$ for $\tau \to \infty$, and $k < k_0$. Since $\vartheta' \geq 0$ for $k < k_0$ and $\vartheta \geq 0$, and $\vartheta' \leq 0$ for $k < k_0$, to prove the assertion formulated, it is sufficient to show that ϑ δ for $\tau \to \infty$, $k \neq k_0$ $(0 < \delta < \infty)$.

We solve the problem with the aid of the real Laplace transform [3]

$$L\vartheta = \int_0^{\infty} e^{-s\tau}\vartheta(\tau)d\tau = f(s).$$

If $\vartheta \to \delta$ for $\tau \to \infty$, it is obvious that

$$\delta/s \leqslant f(s) \quad \text{for } k < k_0, s \geqslant 0 \text{ and}$$
$$0 \leqslant f(s) \quad \text{for } k > k_0, s \geqslant 0. \qquad (7.1)$$

From (5.3) it is easy to obtain an expression for $f(s)$

$$f(s) = \frac{\vartheta_0}{2}\,\frac{acsh + 2(a+b)h + 4}{as^2h + bsh + 2s + c}, \qquad (7.2)$$

where

$$h(s) = \int_0^{\infty} e^{-s\tau}\left(1+\frac{2\tau}{k}\right)^{-1/2 k} d\tau.$$

Let $s \to +0$, then

$$h(s) \sim 1/\sqrt{s}, \; -\ln s, \; 1, \qquad (7.3)$$

for $k = 1, 2, n$ $(n > 2)$, respectively.

For $s \to +0$, from (7.2) and (7.3), we have

$$f(s) \sim \frac{1}{(k_0-k)\sqrt{s}}, \; -\frac{\ln s}{k_0 - k}, \; \frac{1}{k_0-k},$$

for $k = 1, 2, n$ $(n > 2)$, respectively.

From here it follows that the conditions (7.1) are not satisfied already in the proximity of $s = 0$. Hence, $\vartheta \to \infty$ for $k > k_0$, and $\vartheta \to 0$ for $k < k_0$.

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